Notes on Overlapping Generations Models

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Outline

- OLG Endowment Economy
- OLG Production Economy
- Restoring Dynamic Efficiency:
  - Money
  - Social Security

OLG Endowment Economy

Model

- Time: $t = 1, 2, ...$
- Agents: A unit measure of agents are born each period and live for two periods. Generation-t's utility:

$$V_t \equiv u(c_t^t) + \beta u(c_{t+1}^t)$$

- Goods: single *nonstorable* consumption goods.
- Endowment: agent born at t receives endowment y_s^t of goods in period $s \in \{t, t+1\}$.
- Initial old: alive only at date 1, endowmed with y_1^0 and consume c_1^0.
- Market structure: Sequential trading: Goods trade happen in spot market each period. In addition, a one-period risk-free bond is available for saving/borrowing.

Outline of economy

	Time				
Generation	1	2	...	t	$t+1$
$i=0$	(c_1^0, y_1^0)				
$i=1$	(c_1^1, y_1^1)	(c_2^1, y_2^1)			
$i=2$		(c_2^2, y_2^2)	(c_3^2, y_3^2)		
$\vdots$			$\ddots$		
$i=t-1$			$(c_{t-1}^{t-1}, y_{t-1}^{t-1})$	(c_t^{t-1}, y_t^{t-1})	
$i=t$				(c_t^t, y_t^t)	(c_{t+1}^t, y_{t+1}^t)

Table: Structure of OLG economy

Sequential Equilibrium I

A sequential equilibrium (SE) is a set of allocations $(c_1^0, \{c_t^t, c_{t+1}^t, s_t^t\}_{t\geq 1})$ and interest rates $\{r_{t+1}\}_{t\geq 1}$ such that:

- Take r_{t+1} as given, $(c_t^t, c_{t+1}^t, s_t^t)$ solve:

$$\begin{aligned} \max_{\tilde{c}_t^t, \tilde{c}_{t+1}^t, \tilde{s}_t^t} \quad & u(\tilde{c}_t^t) + \beta u(\tilde{c}_{t+1}^t) \\ s.t. \quad & \tilde{c}_t^t + \tilde{s}_t^t = y_t^t \\ & \tilde{c}_{t+1}^t = y_{t+1}^t + (1 + r_{t+1})\tilde{s}_t^t \end{aligned}$$

- Initial old: $c_1^0 = y_1^0$.
- Markets clear:

$$\begin{aligned} c_t^t + c_t^{t-1} &= y_t^t + y_t^{t-1} \\ s_t^t &= (1 + r_t)s_{t-1}^{t-1} \end{aligned}$$

Sequential Equilibrium II

- The asset market clearing is implied by goods market clearing and budget constraints. ▸ Details
- Instead of sequential trade, we can also define another market structure, e.g. the Arrow-Debreu Competitive Equilibrium. ▸ Details
- As in the traditional models, the two market structures yield equivalent equilibria. ▸ Details

No-Trade Result I

Theorem

There is no trade across generations in the barter exchange OLG economy described above. In other words, all agents consume their own endowments (autarky):

$$c_s^t = y_s^t \quad \forall\, t,\ s \in \{t, t+1\}.$$

Proof.

See proof here. ▸ Proof □

No-Trade Result II

- Intuitions:
  - The initial old only lives for one-period, thus they cannot trade with the young at $t = 1$.
  - Since the initial old have to consume their endowment, the initial young have no saving vehicles (because someone's saving must be someone else's borrowing).
  - Thus they arrive at $t = 2$ without any saving, and have to consume their own endowment. So on.
- This is the problem of "double coincidence of wants" in barter economies.
  - In a static framework: I have rice and want to trade for cheese. I can only trade if I find someone who (1) have what I need (cheese) and (2) need what I have (rice).
  - OLG models: initial old only wants to trade within period 1, initial young wants to trade between 1 and 2, period-2 young wants to trade between periods 2 and 3, so on.
  - Thus, by construction, no pair of people satisfy *the coincidence of wants*.

No-Trade Result III

- This no-trade result would break down if (among other reasons):
  - The old is endowed with outside asset (money).
  - The old is endowed with productive capital that accumulates. (Why?)
  - There are more generations: the initial middle-aged may trade with initial young. (See problem set.)
- Conceptually different from the dynamic inefficiency problem (to be discussed next).

Potential T/F/U Questions

1. Consider a barter exchange OLG model (as above) where agents' endowments are stationary: $y_t^t = y^{young}$ and $y_{t+1}^t = y^{old}$ for all cohorts t. Does the model feature trades between generations in the sequential equilibrium?
2. Now suppose there is income growth over time:

$$y_t^t = y_t^{young} = y_0^{young}(1+g)^t$$
$$y_{t+1}^t = y_t^{old} = y_0^{old}(1+g)^t$$

   How would your answer change?

Equilibrium Interest Rate

- The equilibrium interest rate can be backed out from Euler equation:

$$1 + r_{t+1} = \frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} = \frac{u'(y_t^t)}{\beta u'(y_{t+1}^t)}$$

- This is the level of interest rate to induce no saving ($s_t^t = 0$) in equilibrium.
- Potential T/F question: suppose the OLG economy experiences a shock (e.g. automation) that induces a sharper decline in income over the life cycle (i.e. lower y_{t+1}^t / y_t^t). How does it affect (if it does) the interest rate prevailing in the economy? ▸ Solutions

Dynamic Inefficiency I

Theorem

In the OLG economy without population growth, the Sequential Equilibrium is inefficient if and only if $r < 0$. If population grows at rate n, the condition for inefficiency is $r < n$.

Proof.

Consider an income transfer of ε from young to old each period. No trade means consumption of the young (old) reduces (increases) exactly by ε. Of course this makes initial old happy. For cohort $t \geq 1$, change in lifetime utility is

$$\Delta V_t = \left[-u'(c_t^t) + \beta u'(c_{t+1}^t)\right] \varepsilon.$$

This is positive if and only if

$$1 + r_{t+1} = \frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} < 1 \Leftrightarrow r_{t+1} < 0.$$

Dynamic Inefficiency II

- Suppose now population grows at rate n.
- Now take ε from each young (population $(1+n)N_t^{old}$) and transfer $(1+n)\varepsilon$ to each old (population N_t^{old}). Utility change for cohort t:

$$\Delta V_t = \left[-u'(c_t^t) + (1+n)\beta u'(c_{t+1}^t)\right] \varepsilon$$

Transfer is Pareto improving if and only if $\Delta V > 0$, or:

$$1 + r_{t+1} = \frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} = \frac{u'(y_t^t)}{\beta u'(y_{t+1}^t)} < 1 + n$$

or, in other words,

$$r_{t+1} < n.$$

Dynamic Inefficiency III

- The First Welfare Theorem (FWT) states that any competitive equilibrium must be efficient.
- We have shown that the SE of the OLG economy is Pareto inefficient, i.e. the FWT fails here.
- Technically, failure of FWT stems from the fact that total value of endowment can be infinite (whenever $r < n$). ▸ Details
  - More deeply: result of "double infinity" problem (infinite of generations, and infinite of (dated) goods).
  - Good and easy to digest reference: "Notes on the Economics of Infinity" (Shell, JPE 1971).
  - Related to Gamow's Hotel Problem and Hilbert's Grand Hotel Paradox.

Dynamic Inefficiency IV

- Gamow's hotel problem: "An innkeeper has committed each of the denumerably infinite number of beds on a certain rainy night. A guest asks for a bed when all are occupied, but a bed can be found if the innkeeper requires each guest to move down one bed."
- In OLG: $\infty-$generations and $\infty-$commodities mean transfering from young to old can produce extra consumption, like making one extra bed above.
  - Population growth n increases future endowment (relative to past), increasing ability to transfer resources backward, supporting intergenerational transfer.
  - Higher r indicates lesser desire to consume when old of each generation, lowering benefits from transferring, opposing intergenerational transfer.
  - Thus, low r and high n makes room for dynamic inefficiency.

OLG Production Economy

Model

- Time is infinite: $t = 1, 2, ...$
- Utility: generation t agents, population N_t, maximizes:

$$V_t = u(c_t^t) + \beta u(c_{t+1}^t)$$

- Income: young supplies 1 unit of labor (inelastically) and earns w_t. The old do not work and have no labor income.
- Assets: can save in productive capital s_t or bonds a_t:

$$c_t^t + s_t + a_t = w_t$$

$$c_{t+1}^t = (1 + r_{t+1}^k - \delta)s_t + (1 + r_{t+1})a_t$$

- Production: $F(K_t, A_t N_t)$ CRS.
- Markets: all (goods and factors) are competitive.
- Growth: $N_{t+1} = (1+n)N_t$, $A_{t+1} = (1+g)A_t$.

Household Optimization I

- There is no risk, thus no-arbitrage requires

$$r_{t+1} = r^k_{t+1} - \delta$$

- Given prices, (c^t_t, c^t_{t+1}) can be solved from the Euler equation and the intertemporal budget constraint:

$$u'(c^t_t) = \beta(1 + r_{t+1})u'(c^t_{t+1}) \tag{1}$$

$$c^t_t + \frac{c^t_{t+1}}{1 + r_{t+1}} = w_t \tag{2}$$

- Let $s_t = s(w_t, r^k_{t+1})$ be the resulting saving policy function.

Firms

- Firm profit maximization

$$\max_{K_t, N_t} \quad F(K_t, A_t N_t) - r_t^k K_t - w_t N_t$$

- Define $k_t \equiv K_t/(A_t N_t)$ and $f(k_t) \equiv F(K_t, A_t N_t)/(A_t N_t)$.
- Given competitive factors, factor prices are given by marginal products:

$$r_t^k = F_K(K_t, A_t N_t) = F_K\left(\frac{K_t}{A_t N_t}, 1\right) = f'(k_t)$$
$$w_t = F_N(K_t, A_t N_t) = A_t(f(k_t) - f'(k_t)k_t)$$

Market clearing

- Capital supply = capital demand:

$$N_t s(w_t, r_{t+1}) = K_{t+1}$$
$$\Rightarrow \frac{s(w_t, r_{t+1})}{A_{t+1}(1+n)} = k_{t+1}$$

- This implies a law of motion for capital:

$$\frac{s\left(A_t\left(f(k_t) - f'(k_t)k_t\right), f'(k_{t+1})\right))}{A_{t+1}(1+n)} = k_{t+1} \qquad (3)$$

- Generally, the saving function can be complicated.
- Multiple steady states and complicated dynamics are possible.

Tractability I

- A tractable special case for household problem is log utility: $u(c) = \log(c)$.
- Then (1) and (2) imply

$$c_t^t = \frac{1}{1+\beta} w_t, \quad c_{t+1}^t = \frac{\beta(1+r_{t+1})}{1+\beta} w_t$$

- The saving function here is particularly simple:

$$s(w_t, r_{t+1}^k) = \frac{\beta}{1+\beta} w_t$$

- The interest rate r_{t+1}^k does not enter the saving function because with log utility, the wealth and income effect exactly cancels out.
- Useful to keep in mind for future modeling purpose.

Tractability II

- Given $s(w, r) = \frac{\beta}{1+\beta} w$, the law of motion (3) becomes

$$k_{t+1} = \frac{\beta}{1+\beta} \frac{A_t}{A_{t+1}} \frac{f(k_t) - f'(k_t)k_t}{1+n}$$
$$= \frac{\beta}{1+\beta} \frac{f(k_t) - f'(k_t)k_t}{(1+g)(1+n)}$$

- Now assume $F(K, AN) = K^{\alpha}(AN)^{1-\alpha}$, which implies $f(k) = k^{\alpha}$. Thus:

$$k_{t+1} = \frac{\beta}{1+\beta} \frac{(1-\alpha)k_t^{\alpha}}{(1+g)(1+n)} \tag{4}$$

Dynamics similar to Solow model.

Balanced Growth Path

- Let $\widehat{k}$ denotes the solution to (4):

$$\widehat{k} = \left(\frac{\beta}{1+\beta} \frac{1-\alpha}{(1+g)(1+n)} \right)^{\frac{1}{1-\alpha}}$$

- Suppose start from $k_0 = \widehat{k}$, then stays at $\widehat{k}$ forever. Associated interest rate:

$$\widehat{r} = f'(\widehat{k}) - \delta = \frac{1+\beta}{\beta} \frac{\alpha}{1-\alpha} (1+g)(1+n) - \delta$$

- TFP will move $\widehat{w}_t$, $\widehat{c}_t^t$, $\widehat{c}_{t+1}^t$, and $\widehat{s}_t$.

Dynamic Inefficiency I

- Consider the Social Planner's problem:

$$\max \quad u(c_1^0) + \sum_{t=1}^{\infty} \omega_t \left(u(c_t^t) + \beta u(c_{t+1}^t) \right)$$

$$s.t. \quad \frac{c_t^t}{A_t} + \frac{c_t^{t-1}}{A_t(1+n)} + (1+n)(1+g)k_{t+1} = f(k_t) + (1-\delta)k_t$$

- Social Planner's solution:

$$\frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} = f'(k_{t+1}) + 1 - \delta \tag{5}$$

$$\frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} = \frac{\lambda_t}{\lambda_{t+1}}(1+n)(1+g) \tag{6}$$

Dynamic Inefficiency II

- Equation (5) is identical to household.
  - Within generation, the Social Planner chooses allocations exactly as an optimizing individuals would choose.
- But across generations, SP allocations may differ from the competitive equilibrium.
- Compare (6) in the steady state

$$\frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} = (1+n)(1+g) \approx 1+n+g$$

with the Euler equation in competitive equilibrium:

$$\frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} = 1+\widehat{r}$$

Dynamic Inefficiency III

- If $\widehat{r} < n + g$ (e.g. because α very low), then economy is dynamically inefficient.
- What is $n + g$, and why do we compare the competitive equilibrium $\widehat{r}$ with it?
  - Let $c^* \equiv c^y + c^o/(1+n)$ be the aggregate consumption in the steady state. Rewrite the budget constraint:

$$c^* = A\left(f(k^*) - (n+g+\delta)k^*\right)$$

  - Golden saving rule that maximizes aggregate consumption:

$$f'(k_{gold}) = n + g + \delta, \quad r_{gold} \equiv f'(k_{gold}) - \delta = n + g$$

  - If $\widehat{r} < r_{gold}$, the economy *overaccumulates* capital. A reduction of saving can be Pareto improving.

Comparison to Blanchard-Weil

- Weil (1989) argues that finite horizon may not necessarily lead to oversaving.
  - NGM with probability of death p.
  - Possibility of death raises effective discount rate $\rho + p$, which raises MPK, which lowers k^*. Thus: $k^*_{Weil} < k^*_{NGM} < k^*_{gold}$. (The last inequality relies on $\rho > n$)
  - This is in contrary to OLG model, which generates $k^*_{OLG} > k^*_{gold}$ for plausible parameters.
  - Concludes that finite-lives not source of oversaving, but rather the sharp decline in income.
- Barro (1974): introducing altruism removes any finite-lives effects.
- Valid construction of counter-examples.
  - But note that technically, the source of failure of First Welfare Theorem is not finite lives per se, but the infinite number of households.

Restoring Dynamic Efficiency: Social Security

Social Security

- Issue at core of OLG: oversaving.
- Two potential systems of social security programs:
  - Fully-funded system: young makes contribution and gets paid back when old (with interest).
    - Perfectly substitutable with personal saving, thus will have no effect.
  - Pay-as-you-go system (aka unfunded system): contribution of current young directly goes to current old.
    - Transfer of resource from young to old $\longrightarrow$ discourages saving $\longrightarrow$ solve OLG dynamic inefficiency.

Fully-funded Social Security I

- Let d_t denotes the Social Security (SS) contribution of the young at time t. Generation t solves:

$$\max_{c_t^t, c_{t+1}^t, s_t, d_t} \quad u(c_t^t) + \beta u(c_{t+1}^t)$$

$$c_t^t + s_t + d_t = y_t^t$$

$$c_{t+1}^t = y_{t+1}^t + (1 + r_{t+1}^k)(s_t + d_t)$$

- Note that denotes $\widetilde{s}_t \equiv s_t + d_t$, then problem exactly same as in the competitive equilibrium without SS.
- Thus, absent restriction, for any given path of $\{d_t\}_{t\geq 1}$, the households choose:

$$s_t = s_t^{CE} - d_t$$

and consumption are exactly as before.

Fully-funded Social Security II

- Now suppose add in constraint $s_t \geq 0$ i.e. no borrowing.
- Then, if $\{d_t\}_{t\geq 1}$ is such that $d_t \leq s_t^{CE}$ for all t, then still end up at the CE allocations.
- The only time when Fully-funded SS generates allocations that differ from CE is such that $d_t > s_t^{CE}$ for some t.
  - But this means require *more saving* compared to OLG.
  - This cannot be a Pareto improvement.

Unfunded Social Security I

- Let d_t denotes the Social Security (SS) contribution of the young at time t. Generation t solves:

$$\max_{c_t^t, c_{t+1}^t, s_t, d_t} \quad u(c_t^t) + \beta u(c_{t+1}^t)$$

$$c_t^t + s_t + d_t = y_t^t$$

$$c_{t+1}^t = y_{t+1}^t + (1 + r_{t+1}^k)s_t + (1+n)d_{t+1}$$

- Here, with $d_t, d_{t+1} > 0$, lower income when young and higher income when old $\rightarrow$ discourages savings.
- Only s_t goes into capital accumulation, not $s_t + d_t$.
- There exists $\{d_t\}_{t \geq 1}$ that Pareto improves a dynamically inefficient economy.
- But generally, countries have too little capital, not too much.

Restoring Dynamic Efficiency: Money

Outside vs. Inside Money

- Consider hypothetical balance sheets:

Households	
Assets	Liabilities
Bonds $10	
Equity $20	
Money $85	

Firms	
Assets	Liabilities
Money $15	Bonds $10
Machine $15	Equity $20

Government	
Assets	Liabilities
	Money $100

Outside vs. Inside Money

- Bonds and equity are inside money, i.e. assets that are also liabilities of some other agents in the private sector.
- Make sense for households to hold bonds and equity because they are backed by firms' assets, like machine.
- Fiat money is example of outside money, i.e. assets that are not liabilities of anyone *inside* the private sector (hence the term *outside*)

Money

- Outside money is not backed by any assets. Why would anyone hold it?
- Surprising feature of OLG: money *may* have positive value.
  - Medium of exchange function of money.
  - OLG problem: no coincident of wants.
  - Money solves this problem, as long as agents believe in money.
  - There is always a nonmonetary equilibrium: agents do not believe in money, so it is worthless.
- Problem of OLG as a model of money: money circulates too slowly (once a generation).

Model

- Same set up as before, but with money.
- Agent t demands m_t^t money, at price q_t:

$$\begin{aligned} \max \quad & u(c_t^t) + \beta u(c_{t+1}^t) \\ s.t. \quad & c_t^t + s_t^t + q_t m_t^t = y_t^t \\ & c_{t+1}^t = y_{t+1}^t + (1 + r_{t+1}) s_t^t + q_{t+1} m_t^t \end{aligned}$$

- Initial old endowed with amount of money $M \geq 0$.
- Still have Euler equation for bond:

$$u'(c_t^t) = \beta(1 + r_{t+1}) u'(c_{t+1}^t)$$

- Euler equation for money:

$$q_t u'(c_t^t) = \beta q_{t+1} u'(c_{t+1}^t)$$

Nonmonetary equilibrium

- There is always an equilibrium with $q_t = 0$ and the rest of variables identical to before)
- With $q_t = 0$, Euler equation for money trivially satisfied. Any level of holding (including $m_t^t = M$) is consistent with optimality of agents.
- This is the nonmonetary equilibrium. People believe money have no value, so it has no value.

Monetary equilibrium

- Now suppose $q_t > 0$. Combine the two equations:

$$1 + r_{t+1} = \frac{q_{t+1}}{q_t} \left(= \frac{u'(c_t^t)}{\beta u'(c_{t+1}^t)} \right)$$

  In this no risk environment, arbitrage requires that the return on two assets must be the same.

- Let $p_t = 1/q_t$ be the price of consumption good. This is equivalent to:

$$1 + r_{t+1} = \frac{q_{t+1}}{q_t} = \frac{p_t}{p_{t+1}} = \frac{1}{1 + \pi_{t+1}} \Rightarrow r_{t+1} \approx -\pi_{t+1}$$

  This is the Fisher equation for money (which pays zero interest).

Monetary equilibrium features no saving

- As before, we still have $s_t^t = 0$ in this model.
- Proof: sum the time t budget constraints of agent born in t and $t-1$:

$$c_t^t + s_t^t + q_t m_t^t + c_t^{t-1} = y_t^t + y_t^{t-1} + (1+r_t)s_{t-1}^{t-1} + q_t m_{t-1}^{t-1}$$

Market clearing still requires

$$c_t^t + c_t^{t-1} = y_t^t + y_t^{t-1}, \quad m_t^t = m_{t-1}^{t-1} = M$$

which implies

$$s_t^t = (1+r_t)s_{t-1}^{t-1}$$

Given that $s_0^0 = 0$ (initial old has no saving), must be $s_t^t = 0$ for all t.

When does money have positive value?

- Assume $y_t^t = y_1$, $y_{t+1}^t = y_2$. Focus on stationary equilibria: $q_t = q$ for all t.
- Euler equation for money with $q_t = q$:

$$u'(y_1 - qM) = \beta u'(y_2 + qM)$$

- Let $F(\tilde{q}) \equiv u'(y_1 - \tilde{q}M) - \beta u'(y_2 + \tilde{q}M)$. We have $F(0) = r^A \beta u'(y_2)$ and $F'(\tilde{q}) = -Mu''(y_1 - \tilde{q}M) - M\beta u''(y_2 + \tilde{q}M) > 0$.
- Thus $F(\tilde{q}) = 0$ has solution $q > 0$ if and only if $F(0) < 0$ i.e.

$$r^A \equiv \frac{u'(y_1)}{\beta u'(y_2)} < 0$$

Money: intuition

- Money has value when barter exchange economy is inefficient.
- Introducing money restores efficiency since it facilitates resource transfer between generations.
  - With money, the initial old can convince the initial young to give him more consumption. The initial young uses money to convince the next generation to give him consumption, and so on.
  - Again, there is always an equilibrium in which everyone thinks money is just a worthless piece of paper.
  - Infinite time is key. With finite time, money is definitely worthless. (Why?)

Summary

- Introduced the OLG endowment and production economy.
- Both exhibit dynamic inefficiency and failure of first welfare theorem. Problem lies in infinite number of households.
- Dynamic efficiency can be restored either with money or fiscal policy (e.g. unfunded social security programs).

Extra Slides

Arrow-Debreu equilibrium

An Arrow-Debreu equilibrium (ADE) is set of allocations $(c_1^0, \{c_t^t, c_{t+1}^t\}_{t\geq 1})$ and prices $\{p_t\}_{t\geq 1}$ such that:

- Given $\{p_t\}_{t\geq 1}$, (c_t^t, c_{t+1}^t) solve:

$$\max_{\widetilde{c}_t^t, \widetilde{c}_{t+1}^t} \quad u(\widetilde{c}_t) + \beta u(\widetilde{c}_{t+1}^t)$$

$$s.t \quad p_t \widetilde{c}_t^t + p_{t+1} \widetilde{c}_{t+1}^t = p_t y_t^t + p_{t+1} y_{t+1}^t$$

- Initial old consumes his own endowment: $c_1^0 = y_1^0$.
- Markets clear:

$$c_t^t + c_t^{t-1} = y_t^t + y_t^{t-1} \quad \forall\, t \geq 1$$

Equivalence between SE and ADE

- SE and ADE are identical. To see this, consider the budget constraints by SE and ADE:

$$c_t^t + \frac{1}{1+r_{t+1}} c_{t+1}^t = y_t^t + \frac{1}{1+r_{t+1}} y_{t+1}^t$$

$$c_t^t + \frac{p_{t+1}}{p_t} c_{t+1}^t = y_t^t + \frac{p_{t+1}}{p_t} y_{t+1}^t$$

  Given a Sequential Equilibrium, prices $\{p_t\}_{t \geq 1}$ satisfying $p_{t+1}/p_t = (1+r_{t+1})^{-1}$ induce an identical A-D Equilibrium, and vice versa.

- Very intuitive: interest rate is relative price of consumption across time.
- Remarks: by having $s_{t+1}^t = 0$, already implicitly satisfied No-Ponzi ($s_{t+1}^t \leq 0$) and transversality condition ($s_{t+1}^t \geq 0$).

Asset market clearing

- We can prove that asset market clearing is implied by goods market clearing and budget constraint.
- Proof: summing up budget constraints of the current young and current old at time t:

$$\begin{aligned} c_t^t + s_t^t &= y_t^t \\ c_t^{t-1} &= y_t^{t-1} + (1 + r_t) s_{t-1}^{t-1} \end{aligned}$$

and using market clearing

$$c_t^t + c_t^{t-1} = y_t^t + y_t^{t-1}$$

give us

$$s_t^t = (1 + r_t) s_{t-1}^{t-1}.$$

Equilibrium features no trade

- Given that initial old consumes his own endowment $c_1^0 = y_1^0$, market clearing at date 1 implies $c_1^1 = y_1^1$, i.e. the young does not save on date 1: $s_1^1 = 0$. This implies that he has no additional income on date 2, and consumes $c_2^1 = y_2^1$.
- Iterate the argument: $c_s^t = y_s^t$ for all $t \geq 1$ and $s \in \{t, t+1\}$.
- A shorter proof: Asset market clearing:

$$s_t^t = (1 + r_t) s_{t-1}^{t-1}$$

and notes that the initial old has $s_0^0 = 0$. Thus, $s_t^t = 0$ for all generations, implying $c_t^t = y_t^t$ and $c_{t+1}^t = y_{t+1}^t$.

Interest Rate and Growth Rate

- What is the relationship between the growth rate of endowment g and interest rate r in the OLG model?
- Recall that the interest rate is given by

$$1 + r = \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{u'(y_t)}{\beta u'(y_{t+1})}$$

- Assume CRRA utility $u(c) = c^{1-\gamma}$:

$$1 + r_{t+1} = \frac{1}{\beta}\left(\frac{y_t}{y_{t+1}}\right)^{-\gamma} = \frac{1}{\beta}(1+g)^{\gamma}$$

- Take logs, and use the approximation $\ln(1+x) = x$ for $x \approx 0$:

$$r = \rho + \gamma g$$

  with $\rho \equiv -\ln \beta$.
- Same formula as in Neoclassical Growth Model. Higher growth rate g implies higher interest rate.

Failure of First Welfare Theorem I

- Roughly, proof of First Welfare Theorem relies on showing any Pareto-improving allocations must be infeasible.
- Proof by contradition, with last step involving comparing:

$$\sum_{j \in J} p_j x_j' > \sum_{j \in J} p_j \omega_j \tag{7}$$

  where J is the set of commodities.
- The comparison cannot be done if RHS of (7) is infinite, which is precisely the case for the OLG model.
- To see, suppose $y_t^t = 1$, and $y_{t+1}^t = 0$, which implies aggregate endowment each period is $N_t = N_0(1+n)^t$.
- The price of consumption good at date t is $p_t = \frac{1}{(1+r)^t}$.

Failure of First Welfare Theorem II

- RHS of (7) is then:

$$\text{Total endowment value} = N_0 \sum_{t=1}^{\infty} \left(\frac{1+n}{1+r} \right)^t$$

- This sum is convergent (and FWT holds) if and only if

$$\frac{1+n}{1+r} < 1 \Leftrightarrow r > n.$$

- Thus, if $r < n$, the proof of FWT fails.